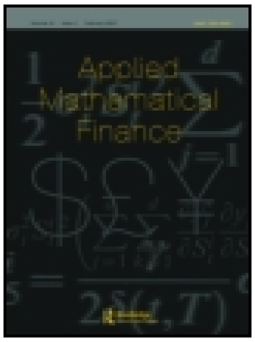

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*Tail behaviour of credit loss distributions for general latent factor models**

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Using a limiting approach to portfolio credit risk, we obtain analytic expressions for the tail behavior of credit losses. To capture the co-movements in defaults over time, we assume that defaults are triggered by a general, possibly non-linear, factor model involving both systematic and idiosyncratic risk factors. The model encompasses default mechanisms in popular models of portfolio credit risk, such as CreditMetrics and CreditRisk$^+$. We show how the tail characteristics of portfolio credit losses depend directly upon the factor model's functional form and the tail properties of the model's risk factors. In many cases the credit loss distribution has a polynomial (rather than exponential) tail. This feature is robust to changes in tail characteristics of the underlying risk factors. Finally, we show that the interaction between portfolio quality and credit loss tail behavior is strikingly different between the CreditMetrics and CreditRisk$^+$ approach to modeling portfolio credit risk.

Keywords: portfolio credit risk, extreme value theory, tail events, tail index, factor models, economic capital, portfolio quality, second-order expansions

JEL Codes: G21, G33, G29, C19

1. Introduction

Management of credit risk is a core function within banks and other lending institutions. There is an extensive literature on how to assess the credit quality of counter-parties in individual loan (or bond)

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transactions, see for example Altman (1983), Caouette *et al.* (1998), and the *Journal of Banking and Finance* (2001, 25(1)) as starting references. In recent years, we have witnessed an increased interest in the modelling and management of *portfolio* credit risk. The portfolio view on credit risk focuses on the probability distribution of potential credit losses for portfolios of loans rather than for individual loans. This requires the consideration of co-movements in loan defaults, i.e. default correlations. In this paper we concentrate on the tail behaviour of portfolio credit losses. Clearly, this is the part of the distribution that both banks and regulators are most concerned about.

Banks usually get into trouble when in a short period of time a substantial part of the loan portfolio deteriorates significantly in quality. This can typically be traced back to some common cause, e.g. a downturn in the economy of a country or region or problems in a particular industry sector, see also Nickell *et al.* (2000) and Bangia *et al.* (2002). Recent examples are the banking problems in Japan, the Asian crisis, and the Russian meltdown. A bank is much less vulnerable to such systematic events if its loan portfolio is well diversified over regions, countries and industries. To evaluate and manage a bank's credit risk, it is therefore not sufficient to scrutinize individual clients to which loans are extended, but also to identify concentration of risks within the portfolio. Portfolio credit risk models allow banks to do just that.

Banks also employ portfolio credit risk models to evaluate activities on a risk/reward basis, using measures such as risk-adjusted return on capital (RAROC) and economic-value-added (EVA); see Matten (2000). Such an evaluation can be done at the level of individual loans or clients, lines of business, or for the bank as a whole. In addition, portfolio credit risk models can be used to evaluate the risks and merits of collateralized loan or bond obligations. A major reason for banks to enter into such structures is to obtain regulatory capital relief. In many cases, however, the majority of the economic risk of the loans involved remains with the issuing bank. A primary motivation for the current review of the 1988 Basel Accord on regulatory capital is to better align regulatory capital requirements with true economic risk. In its latest proposals, the Bank for International Settlements (BIS) has in fact used a portfolio credit risk approach to set risk weights for individual counter-parties (BIS, 2001).

Several models have been put forward in the literature to capture the salient features of portfolio credit risk. The most prominent models are *CreditMetrics* of Gupton *et al.* (1997), *CreditRisk+* of Credit Suisse (1997), *PortfolioManager* of KMV (Kealhofer, 1995), and *CreditPortfolio View* of McKinsey (Wilson, 1997a,b). Despite the apparent differences between these approaches, they exhibit a common underlying framework (Koyluoglu and Hickman, 1998; Gordy, 2000). All models enable the computation or simulation of a probability distribution of credit losses at the portfolio level. The extreme upper quantiles of this distribution are of particular interest.

The explicit relation between model parameters and credit loss tail behaviour is generally badly understood. In the present paper we formulate a general modelling framework encompassing the models mentioned above. In this framework we derive an explicit characterization of the extreme tail behaviour of credit losses in terms of underlying portfolio characteristics. Our approach extends the results in Lucas *et al.* (2001) and contrasts with previous studies of the behaviour of aggregate credit risk. For example, Carey (1998) uses a large database of bonds and a resampling scheme in order to investigate the tail behaviour of credit loss distributions. This approach, however, does not provide an explicit relation between default correlations and credit loss tail behaviour. Moreover, all results are conditional on the extent to which the database used is representative of an actual bond or loan portfolio. Alternatively, Gupton *et al.* (1997) uses an explicit modelling framework and a simulation set-up. The main drawback of a simulation approach is that it is difficult to obtain reliable conclusions regarding tail behaviour, especially if one is concerned with extreme quantiles. Moreover, many different experiments would have to be set up in order to obtain tail properties under a variety of empirically relevant conditions. By contrast, our analytic approach allows for a direct assessment of the relation between default correlations, credit quality, distributional properties, model structure, and credit loss tail behaviour.

Two papers closely related and complementary to our approach are Frey and McNeil (2001, 2002). These authors give a general characterization of different models for dependent defaults. Their focus is on model characterization and identification, especially with regard to the use of default versus asset correlations for model calibration purposes. Our current paper complements these results by taking a more detailed look at the extreme tail behaviour of portfolio credit losses for a specific class of factor models.

In line with the literature, we decompose the risk of an individual loan into a systematic and idiosyncratic risk component. Existing models fully parameterize the distribution of the risk components. For example, CreditMetrics assumes normal risk components, whereas CreditRisk$^+$ assumes gamma distributed components. For our purpose of studying the tail behaviour of portfolio credit losses, however, it suffices to make weak assumptions on the probability of extreme realizations of the risk factors. Put differently, to specify the tail behaviour of portfolio credit losses, we do not need to specify the complete distribution of the underlying risk factors, but only their extreme tail behaviour. This allows for much less restrictive assumptions. In addition, we allow risk factors to be related in a general, possibly non-linear way to a counter-party's creditworthiness. Using statistical Extreme Value Theory, we obtain an expansion of the tail of the credit loss distribution.

Our main contributions to the portfolio credit risk literature are the general modelling framework and the analytic results. It turns out that under quite general conditions credit losses have a polynomial (i.e. fat) rather than an exponential (thin) tail. This polynomial tail can be characterized by a single parameter, the so-called tail index. The tail index specifies the rate of decay in the tail probability. The larger the tail index, the faster the tail probability declines to zero. We show how assumptions on the extreme tail behaviour of the idiosyncratic and systematic risk components determine the extreme credit loss tail behaviour, i.e. the value of the tail index. In particular, we prove that thin tails for idiosyncratic risk and fat tails for systematic risk produce rather unconventional shapes of credit loss densities: they may be actually increasing near the upper end of the support. To the best of our knowledge, such behaviour has not been reported earlier. These rather peculiar density shapes typically contain much more probability mass in the tails compared to a well behaved density function whose tails decline—either exponentially or polynomially—towards the upper end of the credit loss support. Thus, if risk managers do not acknowledge the possibility that their credit portfolio losses may behave like this, they might very well severely underestimate the potential for extreme credit losses.[1] The implication of this result can also be put the other way around. If it is hard to estimate the extreme value behaviour of systematic and idiosyncratic credit risk factors empirically (which is plausible given the quality of data commonly available), care should be taken in interpreting quantile estimates for portfolio credit losses, such as high confidence credit VaRs. In such cases extensive sensitivity analyses

[1] See for example Embrechts *et al.* (1997) or McNeil (1999) for the link between tail behaviour and estimation of risk measures such as credit VaRs, i.e. high confidence quantiles of the credit loss distribution.

of the portfolio results with respect to the tail assumptions of the underlying riskfactors are indispensable in applied work.

We also investigate how credit quality as measured by the probability of default relates to the credit loss tail index. It turns out that credit quality affects the tail behaviour of credit losses differently in the CreditMetrics frame-work compared to CreditRisk$^+$, which are two of the most popular portfolio models to study portfolio credit risk. More specifically, the probability of default directly affects the rate of tail decay as measured by the tail index in the CreditRisk$^+$ model (first-order effect). For the CreditMetrics framework, by contrast, the default probability only affects the scale parameter of the credit losses while leaving the tail index unchanged (second-order effect).

The set-up of the paper is as follows. In Section 2 we provide the basic modelling framework and derive the main results for a homogeneous bond portfolio. We also treat the CreditMetrics and CreditRisk$^+$ models as special cases. The results are generalized in Section 3 to heterogeneous portfolios. Section 4 contains a second-order approach to the tail behaviour of a double Gaussian latent factor model. We highlight the differences between the CreditMetrics and the CreditRisk$^+$ approach regarding the interaction between portfolio credit quality and tail behaviour. Section 5 concludes, while the Appendix gathers all the proofs.

2. Homogeneous bond portfolios

We start our exposition with a very simple portfolio containing n bonds (or loans), each from (to) a different company.[2] The portfolio is homogeneous in the sense that all bonds have the same characteristics. This restrictive setting allows us to derive the main results on the tail behavior of portfolio credit losses. In later sections, we generalize these results to heterogeneous portfolios.

Each bond in the portfolio specifies a future pay-off stream of coupons and/or principal. The value of this stream depends on the creditworthiness of the company issuing the bond. The value of an identical stream of future cash flows will be lower if the company is more likely to default, i.e. has a lower creditworthiness. In our benchmark setting, each company j, where $j=1, \ldots, n$, is characterized by a two-dimensional vector

$$\left(S_j, s^*\right) \tag{1}$$

Here, S_j is a latent variable that triggers a company's default. A prime candidate for S_j is the company's 'surplus' or equity value, i.e. the difference in market value of assets and liabilities, as in the framework of Merton (1974). Other interpretations, however, are also possible; see for example Jarrow and Turnbull (1995) and Duffie and Singleton (1999). If the surplus S_j falls below the threshold s^*, default occurs. As our focus in the present paper is on extreme tail behaviour of credit losses, we concentrate on defaults only and abstract from credit losses due to credit rating migrations, see Gupton *et al.* (1997). Further, for simplicity we set the recovery rate to 0, implying that the loss given default is 100%. This means that in the case of default, the complete amount invested is lost. Alternatively, one can use more realistic values like historical averages of recovery rates. This, however,

[2] We focus on bonds and loans for expositional purposes, but the basic modelling framework remains applicable in case of alternative credit risky securities.

does not affect the rate of tail decay of portfolio credit losses as derived later on. We assume that the initial value of each bond is unity (i.e. each bond values to par at the start). The credit loss on an individual bond j is now given by the random variable

$$1_{\{S_j<s^*\}} \tag{2}$$

where 1_A is the indicator function of the set A.

We assume that S_j obeys the general factor model

$$S_j = g(f, \varepsilon_j) \tag{3}$$

where f is a common factor, ε_j is a firm-specific risk factor, and $g(\cdot, \cdot)$ defines the functional form of the factor model. In this section, we restrict the factor model to be the same for each firm j. This assumption is relaxed in the next section. The formulation in (3) comprises the well-known factor models from the literature. For example, if we set $g(f, \varepsilon_j) = \beta f + \varepsilon_j$ for some factor loading $\beta \in \mathbb{R}$ with Gaussian f and ε_j, we obtain a one-factor version of the CreditMetrics model introduced by Gupton *et al.* (1997). In our present static context, this also coincides with the formulation of CreditPortfolioView of McKinsey (Wilson, 1997a,b). Alternatively, if $g(f, \varepsilon_j) = \varepsilon_j/(\beta f)$ with $\beta > 0$ and ε_j and f exponentially and Gamma distributed, respectively, we obtain the CreditRisk$^+$ specification of Credit Suisse as given in Gordy (2000), compare Credit Suisse (1997).

For sake of simplicity we consider a one-factor version of (3) only. Some results for linear multi-factor models are given in Lucas *et al.* (2001). The key ingredient in (3) is the common risk factor f. The functional dependence of all S_j on this common f induces nonzero asset correlations in the underlying surplus variables that eventually trigger default. Consequently, the model also generates nonzero default correlations. For example, average default rates can be much higher during recessions than during booms, a stylized fact that can be captured by an adequate choice of f.

Given the formulation of the individual credit losses in (2), the credit loss for a portfolio of n loans expressed as a fraction of the amount invested is given by

$$C_n = n^{-1} \sum_{j=1}^{n} 1_{\{S_j<s^*\}} \tag{4}$$

Looking at the extreme tail behaviour of C_n is rather trivial as the support of C_n is discrete. We obtain a continuous credit loss distribution only if we let the number of loans n go to infinity, as in Lucas *et al.* (2001). We follow this approach as it allows us to establish explicit links between the default correlations (as implied by the asset correlation parameter ρ) and credit loss tail thickness.[3] Define

$$C = \lim_{n\to\infty} C_n \tag{5}$$

where the limit exists almost surely, see Theorem 1 further below. Note that a 'standard' central limit theorem to (5) is not applicable due to the common dependence on f for every S_j. As shown in Theorem 1, the limiting credit loss C only depends on the systematic risk factor f and not on the idiosyncratic risk factors ε_j. Using the formulation in (5) rather than (4), we therefore limit the number of stochastic components considerably. This facilitates the study of the tail behaviour of credit losses.

[3] The introduction of stochastic recovery rates can also make the credit loss distribution continuous for finite n, but this would not provide the desired insight into the relation between default correlations and credit loss tail behavior. Indeed, the extreme tail behaviour of portfolio credit losses would be directly equal to the *assumed* tail behaviour for the recovery rates.

As was shown in Lucas *et al.* (2001), empirically relevant quantiles of C_n, e.g. 99% or 99.9%, can be approximated well by quantiles of C, provided the credit portfolio contains at least a few hundred exposures ($n \geq 300$) that are sufficiently granular. These values of n are quite small given the usually large numbers of exposures in typical bank portfolios. Thus we may safely assume that this requirement is satisfied in many situations of empirical interest.

We now introduce our key assumptions on the factor model $g(\cdot,\cdot)$ and the risk factors f and ε_j. For expositional purposes, we again use more restrictive assumptions than necessary. In the discussion of the assumptions, we point out which conditions can be relaxed. Some of these relaxations are worked out in later sections. We use the notation $\bar{F}(x) = 1 - F(x)$, where $F(\cdot)$ is a distribution function.

Assumption 1 (i) $\{\varepsilon_j\}_{j=1}^{\infty}$ is an i.i.d. sequence that is independent of f.

(ii) g is monotonically increasing in both its arguments, such that for all s in the range of g there exist inverse functions $\tilde{\varepsilon}(\cdot,\cdot)$ and $\tilde{f}(\cdot,\cdot)$ defined[4] implicitly by

$$s = g\left(\tilde{f}(s, \varepsilon), \varepsilon\right) = g\left(f, \tilde{\varepsilon}(f, s)\right)$$

(iii) The supports of ε_j and f are unbounded to the right and left, respectively. Furthermore, for all s we have $\lim_{x\uparrow\infty} \tilde{f}(s, x) = -\infty$, and $\lim_{x\downarrow -\infty} \tilde{\varepsilon}(x, s) = \infty$.

Part (i) of the assumption is standard. The identically distributed requirement is less crucial and will be relaxed in the next section. Part (ii) of Assumption 1 requires the factor model to be increasing in the risk factors. The focus on *increasing* g is not very restrictive *per se*. For example, the specification of CreditRisk$^+$ ($S_j = \varepsilon_j/(\beta f)$) does not satisfy the assumption directly, as it is decreasing in f. If this is the case, however, we can usually easily transform variables and consider $g(f^*, \varepsilon_j)$ with $f^* = -f$, which is increasing in f^*. The additional condition in part (ii) requires invertibility of the factor model g. The inverses must be well defined and lie in the appropriate supports of the original risk factors. In particular, we only consider factor models from which we can always uniquely retrieve an element from the vector (S_j, f, ε_j) given the other two elements. Note that both the linear CreditMetrics model ($S_j = \beta f + \varepsilon_j$ with S_j, f, and ε_j in $\mathbb{R}$) and the multiplicative CreditRisk$^+$ model ($S_j = \varepsilon_j/(\beta f)$ with S_j, f, and ε_j in $\mathbb{R}^+$) satisfy this criterion. Part (iii) states that the supports of f and the ε_j are unbounded from below and above, respectively. This assumption is not crucial, but greatly simplifies subsequent notation. Again, there is no loss in generality as situations with a bounded support can be accommodated by an appropriate change of variables. The last part of condition (iii) has the following intuition. Consider the borderline case where a firm j is almost pushed into bankruptcy. If common risk factors (f), e.g. the state of the business cycle, are extremely adverse, then firm-specific conditions (ε_j) have to be extremely favorable to prevent the firm from going bankrupt. We thus exclude bankruptcies that are solely induced by adverse values of f regardless of firm-specific risk ε_j (or vice versa).

A second set of assumptions constrains the different types of tail behaviour for the risk factors f and ε_j. In studying the tail behaviour of aggregate credit losses we will either start from polynomially declining tails for the underlying risk components (Assumption 2A) or exponentially declining tails

[4] Note that one should carefully distinguish between the random variables f and ε and the implicit functions $\tilde{f}(\cdot,\cdot)$ and $\tilde{\varepsilon}(\cdot,\cdot)$.

(Assumption 2B). Let $F(\cdot)$ and $G(\cdot)$ denote the (almost everywhere continuously differentiable) distribution functions of ε_j and f, respectively.

Assumption 2A (i) Let $F(\cdot)$ denote the (almost everywhere continuously differentiable) distribution function of ε_j. Then $F(\cdot)$ has a right-hand tail expansion of the form

$$\bar{F}(x) = x^{-\nu} \cdot L_1(x) \tag{6}$$

where $L_1(\cdot)$ is a slowly varying function for $x \to \infty$.

Similarly, let $G(\cdot)$ denote the distribution function of f. Then $G(\cdot)$ has a left-hand tail expansion of the form

$$G(x) = (-x)^{-\mu} \cdot L_2(x) \tag{7}$$

with $L_2(\cdot)$ a slowly varying function for $x \to -\infty$.

(ii) The function $x \mapsto -\tilde{f}(s^*, x)$ is regularly varying at infinity with index $\zeta_1 > 0$, so $-\tilde{f}(s^*, x) = x^{\zeta_1} L_f(x)$, with L_f slowly varying.

Note that ν_1 and μ_1 can be interpreted as the tail indices of the risk factors ε_j and f, respectively. Assumptions 2A places further restrictions on the stochastic behaviour of f and ε_j and on the factor model g. It states that f and ε_j have polynomial left-hand and right-hand tails, respectively. Note that we only make assumptions about the extreme tail behaviour of these random variables. By contrast, both CreditMetrics and CreditRisk$^+$ make much more restrictive assumptions by specifying the complete stochastic behaviour of the risk factors. Using part (i) of Assumption 2A, we allow for any tail shape that lies in the domain of attraction of a Fréchet (or a Weibull) law (Embrechts *et al.*, 1997). An example of this is a distribution with polynomial tails, e.g. the Student t distribution. Part (ii) of Assumption 2A further limits the number of allowed factor model specifications. For example the specification $g(f, \varepsilon_j) = \varepsilon_j \exp(f)$ is not allowed as it is 'not balanced' in f and ε_j. Again, such unbalancedness can usually be resolved by an appropriate change of variables.

To state the appropriate conditions for exponential rather than polynomial tails, we introduce the class of functions $\mathcal{M}_a(\theta)$.

Definition 1 The class $\mathcal{M}_a(\theta)$ for $a \in [-\infty, +\infty]$ and $\theta \in \mathbb{R}$ consists of measurable functions $\chi : \mathbb{R} \to \mathbb{R}$ such that

(i) $\lim_{x \to a} \chi(x)/x^{\theta}$ exists and is finite;

(ii) $\lim_{x \to a} x\chi'(x)/\chi(x) = \theta$.

We will only use the cases $a = 0$ and $a = \pm\infty$ for $\mathcal{M}_a(\theta)$. Notice that $\chi \in \mathcal{M}_a(\theta)$ if and only if $x \mapsto \chi(x)/x^{\theta} \in \mathcal{M}_a(0)$. If a function χ is regularly varying at infinity with index θ and satisfies that for some real number B we have $\chi(x) = \chi(B) + \int_B^x \chi'(t)\mathrm{d}t$ for all $x > a$ with χ' ultimately monotone, then it follows from Theorem 2.4 of Seneta (1976) that $x\chi'(x)/\chi(x) \to \theta$ for $x \to \infty$. But then, we also have $\chi \in \mathcal{M}_\infty(\theta)$ under the additional condition that $\lim_{x \to \infty} \chi(x)/x^{\theta}$ exists and is finite. Also notice also if $\chi(x) = x^{\theta} L(x)$, then $\chi \in \mathcal{M}_\infty(\theta)$ if and only if $L \in \mathcal{M}_\infty(0)$. Imposing the condition that a function χ belongs to $\mathcal{M}_\infty(\theta)$ can be viewed as strengthening the representation of regularly varying functions in the sense that we can now write $\chi(x) = c_B x^{\theta} \exp\left(\int_B^x \frac{\chi(u)}{u} \mathrm{d}u\right)$ for all x greater than or equal to some number B, where $\lim_{x \to \infty} \chi(x) = 0$, see Theorem 1.2 in Seneta (1976) and equation (1.5.1) in Bingham *et al.* (1987). It will prove later that this class of functions is very useful for focusing on tail behaviour.

Assumption 2B (i) As opposed to Assumption 2A, the right-hand tail expansion of $F(\cdot)$ has the form

$$\bar{F}(x) = \exp(v_1 x^{v_2}(1+\chi(x))) \tag{8}$$

with $v_1 < 0$, $v_2 > 0$, $\lim_{x\to\infty} \chi(x) = 0$ and $1+\chi \in \mathcal{M}_\infty(0)$. Similarly, the left-hand tail expansion of $G(\cdot)$ has the form

$$G(x) = \exp(\mu_1(-x)^{\mu_2}(1+\xi(x))) \tag{9}$$

with $\mu_1 < 0$, $\mu_2 > 0$, where we also require that $1+\xi$ belongs to $\mathcal{M}_{-\infty}(0)$ with $\lim_{x\to-\infty} \xi(x) = 0$.

(ii) $\tilde{f}(s^*, y) = -\zeta_2^{1/\mu_2} y^{v_2/\mu_2}(1+\eta(y))$, with $1+\eta \in \mathcal{M}_\infty(0)$ and $\lim_{y\to\infty} \eta(y) = 0$.

Assumption 2B resembles Assumption 2A except for the fact that we now have exponential rather than polynomial tails. Notice that Assumption 2B imposes more stringent conditions on the various tails than Assumption 2A. Not only do we assume that the slowly varying functions in (the exponents of) the tail expansions in fact have a limit, but we restrict the speed of convergence by, e.g. $y\eta'(y) \to 0$ for $y \to \infty$. Though our formulation is not as general as that in Theorem 3.3.26 of Embrechts *et al.* (1997), we still cover a wide range of distributions that are commonly used in empirical exercises, e.g. the normal and the gamma distributions of CreditMetrics and CreditRisk$^+$, respectively. Part (ii) of Assumption 2B is a modified balancedness condition, similar to part (ii) of Assumption 2A.

Assumptions 2A and 2B are easily applied to the standard credit risk models as well as to straightforward extensions of these. We do this later in the paper by giving explicit examples. The following theorem follows directly from Williams (1991), Theorem 12.13. For completeness, its proof is given in the Appendix.

Theorem 1 Given Assumption 1 and C as defined in (5), we have that $C = \lim_{n\to\infty} C_n$ exists a.s. and

$$C = P\left[S_j < s^* \middle| f\right] \tag{10}$$

Note that C is still stochastic due to its dependence on f. We now study the extreme tail behaviour of portfolio credit losses C. The following theorems are proved in the Appendix.

Theorem 2 Let H be the distribution function of C. Given Assumptions 1 and 2A, C lies in the maximum domain of attraction of the Weibull with tail index

$$\alpha = \zeta_1 \mu_1 / v_1$$

meaning that

$$1 - H(c) = (1-c)^\alpha \cdot L(1/(1-c)) \tag{11}$$

with c tending to the maximum credit loss 1.

Theorem 2 directly reveals the extreme tail behaviour of credit losses. In particular, the fact that C lies in the domain of attraction of the Weibull distribution implies that the distribution $H(\cdot)$ of C has the form given in (11). The theorem further reveals how the tail index of the credit loss distribution (α) depends on the tail indices of the latent factors (f and ε_j) and on the factor model g. The dependence on the factor model enters through ζ_1, which is controlled by the balancedness condition (ii) in Assumption 2A. If the tails of f and ε_j are both of the Fréchet type (Embrechts *et al.*, 1997), the theorem shows that the tail index of the credit loss distribution is directly proportional to the ratio of the tail index of f to that of ε_j. The tail index of C can thus be very small provided ν_1 is much larger than μ_1. Put differently, the tails of the credit loss distribution may be very fat if the idiosyncratic risk factor is much lighter tailed than the systematic risk factor. This has a straightforward economic interpretation. If the tail of the common risk factor f is heavier than the tail of the idiosyncratic risk factor ε_j, extreme falls in the variables S_j triggering default will primarily be induced by bad realizations of f. Consequently, it is more likely that a large number of bonds in the portfolio default simultaneously (due to extremely adverse common shocks) rather than separately (due to extremely adverse idiosyncratic shocks). This clustering effect in individual defaults increases the likelihood of extreme portfolio losses and corresponds with a slower rate of tail decay compared to the combination of thin-tailed common and heavy-tailed idiosyncratic shocks.

We obtain a similar theorem for the case of exponential tails.

Theorem 3 Given Assumptions 1 and 2B, C lies in the maximum domain of attraction of a Weibull with tail index

$$\alpha = \zeta_2 \mu_1 / \nu_1$$

An interesting implication of this theorem is that the tail index of credit losses can be finite even if the underlying risk factors f and ε_j are both thin-tailed, see also Lucas *et al.* (2001) and Figure 1 below.

To illustrate this result, consider two examples: the CreditMetrics model of Gupton *et al.* (1997), and the CreditRisk$^+$ model of CreditSuisse as modified by Gordy (2000). First, consider the linear factor model of CreditMetrics, $S_j = \beta f + \varepsilon_j$, with f and ε_j both standard normally distributed and $\beta > 0$, such that Assumption 2B applies. We thus obtain $\nu_1 = \mu_1 = -1/2$, and $\nu_2 = \mu_2 = 2$. From the factor model inversion $\tilde{f}(s, \varepsilon) = (s - \varepsilon)\big/\beta$ it follows that $\zeta_2 = \beta^{-2}$ and, thus, $\alpha = \beta^{-2}$. This confirms the results in Lucas *et al.* (2001). A higher systematic risk component (i.e. higher β) transforms into a lower tail index of C, which implies that more systematic risk results in fatter tails for portfolio credit losses. For CreditRisk$^+$, the factor model reads $S_j = \varepsilon_j/(-\beta f)$, where ε_j is standard exponentially distributed, and $(-f)$ has a gamma distribution with parameters γ_1 and γ_2. We have $\nu_1 = -1$ and $\nu_2 = 1$ for the exponential, and $\mu_1 = -1/\gamma_2$, $\mu_2 = 1$, and $\xi(y) = \gamma_2(1 - \gamma_1)\ln(y)/y$ for the gamma (see Abramowitz and Stegun, 1970 equation 6.5.32). It is easily checked that $\xi(y) \in \mathcal{M}_\infty(0)$. Furthermore, inverting the factor model gives $\tilde{f}(s, \varepsilon) = \varepsilon/(-\beta s)$, such that $\zeta_2 = (\beta s^*)^{-1}$. Therefore, following Theorem 3 the tail index of portfolio credit losses is given by $\alpha = (\beta s^* \gamma_2)^{-1}$. Just as in the CreditMetrics specification, we see that for the CreditRisk$^+$ specification a more dominant common risk component (higher β) results in a lower rate of tail decline. In contrast to the CreditMetrics model, however, we also see that the portfolio quality enters the tail index. This quality is measured by the magnitude of the default threshold s^*, which for this model specification is strictly positive. Portfolios with a higher quality level will have a lower value for s^*, and thus a higher tail index. In Section 4 we prove that also the CreditMetrics model is affected by portfolio quality. In contrast to the CreditRisk$^+$ specification where the effect is of first-order, portfolio quality only has a second-order effect in the CreditMetrics specification (i.e. only affects the slowly varying function and not the tail index). This provides yet another difference between the two modelling frameworks, see also Gordy (2000).

A graphical illustration of the analytic results in Theorems 2 and 3 is contained in Figure 1. The figure presents credit loss densities for the linear factor model of CreditMetrics, slightly

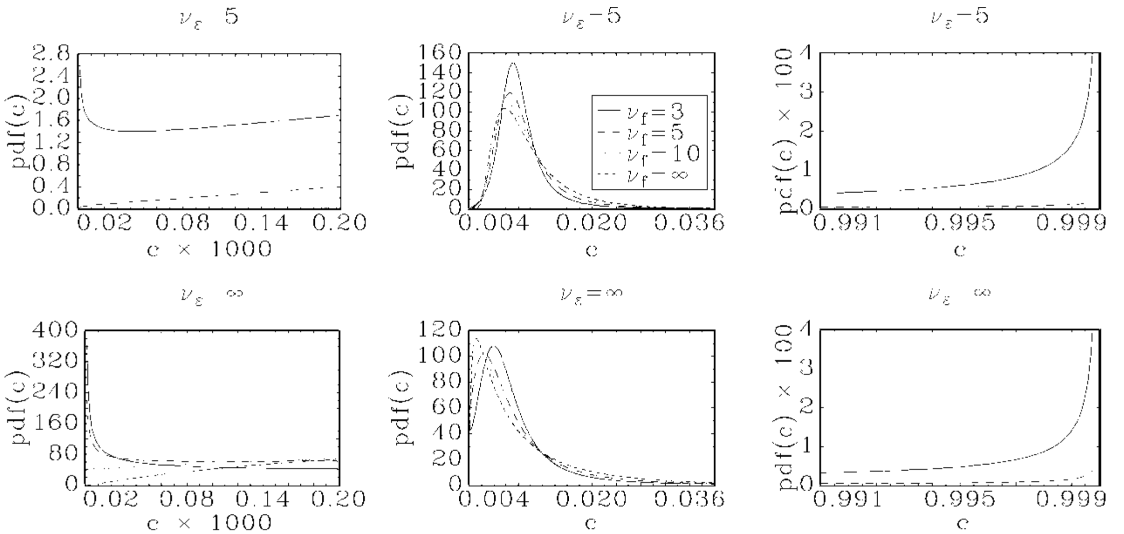

Fig. 1. Credit loss distributions with different tail indices. The figure contains the credit loss densities for a homogeneous portfolio. The underlying factor model is linear, $g(f,\ \varepsilon_j)=a\cdot f+b\cdot\varepsilon_j$, with $a=\rho(1-2/\mu_1)^{1/2}$, $b=\left(1-\rho^2\right)^{1/2}(1-2/v_1)^{1/2}$, and $\rho=0.15$. All ε_j are identically distributed. The risk factors f and ε_j both follow a standardized Student t distribution with μ_1 and v_1 degrees of freedom, respectively. The default probability is 1%. The left-hand plots display the credit portfolio loss density's behaviour in the extreme left-hand tail. The middle plots display the behaviour in the middle of the support, and the right-hand plots give the extreme right-hand tail behaviour. Note the different scaling of the axes, especially the horizontal axis in the left-hand plots and the vertical axis in the right-hand plots.

reparameterized as

$$S_j=\rho(1-2/\mu_1)^{1/2}f+\left[\left(1-\rho^2\right)(1-2/v_1)\right]^{1/2}\varepsilon_j \tag{12}$$

For illustration purposes, we set $\rho=0.15$. Results are similar for other values of ρ between 0 and 1. We further assume that f and ε_j follow a Student t distribution with degrees of freedom μ_1 and v_1, respectively. Note that the rescaled risk factors $(1-2/\mu_1)^{1/2}f$ and $(1-2/v_1)^{1/2}\varepsilon_j$ now both have zero mean and unit variance, as is common in the CreditMetrics framework. We set the probability of default to 1%. The resulting credit loss densities are given in Figure 1 over various relevant regions of the domain $C\in[0,\ 1]$. If $v_1,\ \mu_1<\infty$, Theorem 2 applies, such that the tail index of C is given by $\alpha=\mu_1/v_1$. If $v_1,\ \mu_1\uparrow\infty$, risk factors are normally distributed and the tail index of C is given by $\alpha=(1-\rho^2)/\rho^2$ (Lucas *et al.*, 2001).

The first thing to note in Figure 1 are the middle plots. These reveal the typical shape of credit loss distributions known in the literature. Due to the common dependence on f, defaults are correlated. This in turn gives rise to a portfolio credit loss density that is right-skewed and has a fat right-hand tail. More peculiar are the steeply decreasing and increasing shapes of the density in the extreme left-hand (see left-hand plots) and right-hand tail (see right-hand plots), respectively. These characteristics only show up in the plots if either the density of f or ε_j has polynomial rather than exponential tails. This is due to the specific value of ρ chosen. If $\rho^2 > 0.5$, similar patterns can show up if both tails are of the exponential type, e.g. normal. As the assumption of thin tails for f and ε_j has been predominant in the literature, it is not surprising that these unconventional shapes of the credit loss density have not been considered earlier. The peculiar shape of the densities can be understood as follows. Situations in which all firms default together or do not default at all correspond to extremely negative and positive realizations, respectively, of the systematic factor f. These situations are more likely to occur if the distribution of f exhibits heavier tails than the idiosyncratic risk factor ε_j, because then extremely bad realizations of f are less likely to be offset by extremely good realizations of ε_j.

The phenomena displayed in Figure 1 can also be illustrated using the analytical expression of the credit loss density. From the proof of Theorem 2 in the Appendix, it follows that for a linear factor model $S_j = af + b\varepsilon_j$, this density $H'(c)$ has the form

$$H'(c) = \frac{b}{a} \cdot \frac{G'\left(\frac{s^*}{a} - \frac{b}{a} F^{-1}(c)\right)}{F'(F^{-1}(c))} \tag{13}$$

where F', G', and H' are the derivatives of the distribution functions F, G, and H, respectively. If the tails of f are lighter than those of ε_j, the numerator tends faster to zero for c tending to either 0 or 1 (and thus $F^{-1}(\cdot)$ tending to $-\infty$ or $+\infty$). By contrast, if the tails of ε_j are lighter, the denominator tends to zero at a faster rate. As a consequence, the density diverges to ∞ for both $c \downarrow 0$ and $c \uparrow 1$. If both tails are equally heavy, (13) shows that what matters at the extremes of the support is the size of b/a. For example, for polynomial tails of f and ε_j that have the same tail index, it follows from (13) that the density tends to a non-zero limit at the edge of its support if $|b| < |a|$.

The results so far also have a practical edge for credit risk management. The likelihood of extreme credit losses is increased if the common risk factor has fatter tails than the idiosyncratic risk factor. As it is generally difficult to reliably estimate the tail-fatness of f and ε_j from the empirical data that are typically available, a more conservative approach than that based on normally distributed risk factors can be warranted for prudent risk management. Especially in the upper quantiles of the credit loss distribution, more probability mass might be concentrated than suggested by the normality assumption for common and idiosyncratic risk (see also the numerical results in Lucas *et al.*, 2001).

3. Heterogeneous bond portfolios

So far we considered the portfolio credit loss distribution for homogeneous portfolios and a one-factor model governing defaults. We now extend the results to heterogeneous portfolios consisting of m homogeneous groups. We use i as the index of group i, $i = 1, \ldots, m$. Each group consists of $n_i = n_i(n)$ companies with $\sum_{i=1}^{m} n_i = n$. Notice also that for each company j there exists exactly one $i = i_j$ such that this company belongs to group i. We now have a company/group specific factor model, such that for all $j = 1, \ldots, n$ it holds that

$$S_j = g_i(f, \varepsilon_j)$$

for some $i=1, \ldots, m$. We modify the assumptions from Section 2 accordingly. In order to avoid uninteresting pathological situations in the present context we also make the assumption that the relative sizes of the groups $\lambda_i(n) = \frac{n_i(n)}{n}$ eventually stabilize. That is $\lambda_i = \lim_{n \to \infty} \lambda_i(n)$ is assumed to exist for all i.

Assumption 1′ The same as Assumption 1, except for the following modifications:

(i) The ε_j are still independent and are within each group identically distributed. The common distribution function in group $\imath$ is denoted by F_i.

(ii) The factor models g_i are increasing in both arguments and the inverse functions $\tilde{f}_i(s, \varepsilon)$ and $\tilde{\varepsilon}_i(f, s)$ exist and are well defined for all s, ε, f in their relevant supports.

(iii) Unchanged.

(iv) There exists an index $\imath \in \{1, \ldots, m\}$ and a constant K such that

$$\lim_{x \to -\infty} \frac{\sum_{i=1}^{m} \lambda_i \left(1 - F_i\left(\tilde{\varepsilon}_i\left(x, s_i^*\right)\right)\right)}{1 - F_\imath\left(\tilde{\varepsilon}_\imath\left(x, s_\imath^*\right)\right)} = K > 0 \tag{14}$$

Assumption 2A′ Similar to Assumption 2A, except:

(i) Each F_i has a right-hand tail expansion as in (6), but with parameter ν_{1i}.

(ii) The function $x \mapsto -\tilde{f}_\imath(s^*, x)$ is regularly varying at infinity with index $\zeta_1 > 0$, so $-\tilde{f}_\imath(s^*, x) = x^{\zeta_1} L_f(x)$, with L_f slowly varying.

Assumption 2B′ Similar to Assumption 2B, except:

(i) Each F_i has a right-hand tail expansion as in (6), but with parameters ν_{1i}, ν_{2i}, and ν_{3i}.

(ii) $\tilde{f}_\imath(s^*, y) = -\zeta_2^{1/\mu_2} \imath y^{\nu_2/\mu_2}(1 + \eta(y))$, with $1 + \eta \in \mathcal{M}_\infty(0)$ and $\lim_{y \to \infty} \eta(y) = 0$.

The main relaxations with respect to the previous set of assumptions concern the group-specific factor models and distributions of the idiosyncratic risk components. Also note that the credit quality as measured by s_i^* may differ across groups.

Assumption 1′ on the factor models all being increasing in f is more restrictive for heterogeneous portfolios than for homogeneous portfolios. In particular, it is no longer always possible to meet this assumption by an appropriate change of variables. As an example, consider two groups where one has a factor model that is increasing in f, while the other factor model is decreasing in f. By changing variables from f to f^* to make the latter model increasing in f^*, one makes the former model decreasing in f^*. Such situations are, however, of limited practical interest as they imply both positive and negative correlation between companies' surplus variables and macroeconomic conditions for significant parts of the portfolio. Empirical work shows that these correlations are predominantly positive, see Das *et al.* (2002).

Part (iv) of Assumptions 1′ is new and requires that for extreme common risk factor realizations f one of the idiosyncratic tails dominates the other tails. Parts (ii) of Assumptions 2A′ and 2B′ now only need to be satisfied for group $\imath$ rather than for every group $i=1, \ldots, m$. Note that the limit in (iv) exists if for all i

$$\ell_i = \lim_{x \to -\infty} \frac{\lambda_i\left(1 - F_i\left(\tilde{\varepsilon}_i\left(x, s_i^*\right)\right)\right)}{1 - F_\imath\left(\tilde{\varepsilon}_\imath\left(x, s_\imath^*\right)\right)}$$

exists and is finite. In that case we have $K = \sum_{i=1}^{n} \lambda_i \ell_i$.

The assertion of Theorem 1 now takes a different form, which can be proved similarly. Different from (10) we have the following formulation of portfolio credit losses:

$$C = \sum_{i=1}^{m} \lambda_i \cdot P\left[g_i(f, \varepsilon_i) < s_i^* \middle| f\right] = \sum_{i=1}^{m} \lambda_i \cdot F_i\left(\tilde{\varepsilon}_i\left(f, s_i^*\right)\right) \tag{15}$$

where we have replaced the firm index j of ε and $\tilde{\varepsilon}$ by the group index i. For each firm in group i, ε_i follows the distribution F_i, and the ε_i are independent. The constants s_i^* determine the default probability in group i. As said before, the constants λ_i denote the (asymptotic) relative size of group i. Alternatively, one can allow for different loan sizes or recovery rates between groups and incorporate these in λ_i. This does not affect the rate of tail decay, but may impact the upper endpoint of the support of C. For simplicity, we do not consider this case here.

As can be seen from (15), only the groups with a positive λ_i contribute to the asymptotic credit loss. We now discard all bonds in group i' for which $\lambda_{i'} = 0$. The resulting portfolio now contains $n' = n\text{-}n_{i'}(n)$ bonds and the relative sizes of the groups become $\lambda_{i'}(n') = \frac{n_i(n')}{n'}$. It is, however, fairly easy to see that still $\lim_{n' \to \infty} \lambda_{i'}(n') = \lambda_i$. Therefore Equation 15 is still valid for the smaller portfolio, since for the original portfolio the i'-th group contributed nothing to the asymptotic credit loss. Henceforth we assume a portfolio for which all λ_is are strictly positive.

We have the following theorem on the tail index of credit losses for heterogeneous portfolios. The theorem is proved in the Appendix.

Theorem 4 Let Assumptions 1′ and 2A′ be satisfied, then C lies in the maximum domain of attraction of the Weibull with tail index

$$\alpha = \zeta_1 \mu_1 / \nu_{1\imath}$$

We obtain a similar theorem for exponential tails.

Theorem 5 Let Assumptions 1′ and 2B′ be satisfied, then C lies in the maximum domain of attraction of the Weibull with tail index

$$\alpha = \zeta_2 \mu_1 / \nu_{1\imath}$$

An important implication of Theorems 4 and 5 is that in order to characterize the extreme tail behaviour of portfolio credit losses, we do not have to take the complete portfolio into account. Only segment $\imath$ is important to compute the tail index. In fact, the tail index is the same for a heterogeneous portfolio compared to a homogeneous portfolio of the same size consisting of loans to group $\imath$ only. This immediately follows from the fact that the size of the investment in group $\imath$ ($\lambda_\imath$), does not enter the expression for the tail index. To provide some further insight, we focus on the definition of $\imath$. Assume a factor model that is identical across groups, $g_i(f, \varepsilon) \equiv g(f, \varepsilon)$, but with the idiosyncratic risk factors still allowed to have different distributions across groups. According to (14), $\imath$ characterizes the group that has the thickest right-hand tails for the idiosyncratic risk component. Thus, the group with the heaviest idiosyncratic tail dictates portfolio credit loss tail behaviour. In particular, the heavier this tail compared to the tail of f, the lighter the tail of portfolio credit losses C; see also Frey and McNeil (2001). The intuition for this result follows from the limiting approach taken. Idiosyncratic risk is diversifiable and therefore not incorporated in C, which only depends on common risk f. If a part of the portfolio has a strong idiosyncratic risk component, this part of the portfolio is less likely to be pushed into default by movements in common risk only. In the extreme right-hand tail of credit losses, all bonds in the portfolio have to default due to adverse common risk realizations only. As argued, the most problematic cases in this respect are precisely the bonds in group ι, which are more easily pushed into default by idiosyncratic risk compared to common risk. Therefore, this group entirely determines the tail behavior near the maximum credit loss.

4. Second-order tail expansion

In Section 2, we showed that the tail index of credit losses is only influenced by portfolio quality s^* in the CreditRisk$^+$ specification of Gordy (2000), and not in the CreditMetrics framework. In the present section, we prove that credit quality does also influence the tail behaviour of credit losses in the CreditMetrics framework, but through a different channel. We again focus on a homogeneous portfolio and the linear factor model $S_j = \rho f + (1-\rho^2)^{1/2}\varepsilon_j$ with Gaussian risk factors. In order to study the impact of changes in credit quality on portfolio credit losses in a CreditMetrics framework, we consider a second-order tail approximation. In particular, we derive an expression for the slowly varying function $L(\cdot)$ in (11) that is correct up to first order. In the Appendix, we prove the following theorem.

Theorem 6 Given the homogeneous Gaussian linear factor model setting

$$S_j = \rho f + \sqrt{1-\rho^2}\,\varepsilon_j$$

for $\rho \in [-1, 1]$, the distribution of C has a tail expansion for $c \uparrow 1$ of the form

$$P[C > c] = (1-c)^{(1-\rho^2)/\rho^2} \cdot L(1/(1-c)) \tag{16}$$

where L is a function that is slowly varying at infinity and that satisfies

$$L(x) = \frac{\rho\left(\ln\left(x^2\right)\right)^{\frac{1-3\rho^2}{2\rho^2}}}{\sqrt{1-\rho^2}} \exp\left[-\frac{(s^*)^2}{2\rho^2} + \frac{s^*\sqrt{\ln(x^2)}\sqrt{1-\rho^2}}{2\rho^2}\right] \cdot (1+o(1)) \tag{17}$$

The theorem gives a more explicit form of the slowly varying function $L(\cdot)$ in the tail expansion. Gathering the components of $L(x)$ that depend on x, we have

$$L(x) \propto \exp\left[\frac{1-3\rho^2}{2\rho^2}\ln\left(\ln\left(x^2\right)\right) + \frac{s^*\sqrt{\ln(x^2)}\sqrt{1-\rho^2}}{2\rho^2}\right] \tag{18}$$

The dominant term in $L(x)$ as a function of $x \uparrow \infty$ is therefore

$$\exp\left[\frac{s^* \sqrt{\ln(x^2)}\sqrt{1-\rho^2}}{2\rho^2}\right] \tag{19}$$

First note that $s^* = \Phi^{-1}(p)$ for a default probability p. For p less than 50%, the default threshold s^* will be negative. Moreover, s^* is increasing in p. If $s^* < 0$, (19) is decreasing in x, because $\rho^2 \leq 1$. The smaller the default probability p, the faster the rate of decline of (19) in x. A higher level of portfolio quality, i.e. a lower p and more negative s^*, increases the rate of tail decline for credit losses. Therefore, less far out in the credit loss tail, tails may appear thinner than suggested by the result in Theorem 3. This effect, however, is only of second order. In the extreme tail, the slowly varying function is again dominated by the factor $(1-c)^{(1-\rho^2)/\rho^2}$ in (16). This contrasts with the finding for the CreditRisk$^+$ model in Section 2, where s^* entered the tail index of credit losses directly.

5. Concluding remarks

In this paper, we followed a limiting approach to determining the distribution of aggregate portfolio credit risk. Using a general (nonlinear) latent factor model, we decomposed credit risk into a systematic and an idiosyncratic risk factor. The model allows for different rates of tail decay for the underlying risk components. We proved that under general conditions the distribution of portfolio credit losses exhibits a polynomially decaying tail. This is important for credit risk management.

We showed that the tail index of credit losses for homogeneous portfolios directly relates to the tail indices of the systematic and idiosyncratic risk components, and to the functional specification of the factor model. The results were illustrated by computing the tail decay rate of aggregate credit losses for the CreditMetrics and CreditRisk$^+$ models, two of the most common credit risk portfolio models available in the literature. This revealed a striking difference: the portfolio quality has a first-order effect on the rate of tail decline under the specification of CreditRisk$^+$ as given in Gordy (2000), but not under that of CreditMetrics.

Moreover, we showed that the tail index of portfolio credit losses is very small if the tail of the systematic risk component is much heavier than the tail of the idiosyncratic risk component. In particular, the density of credit losses may then be *increasing* towards the edges of its support. This means that extreme credit losses may show up with a much larger probability than suspected on the basis of a factor model with Gaussian systematic and idiosyncratic risk.

We generalized our analytical results to a heterogeneous portfolio set-up by allowing distributions of idiosyncratic risk, default probabilities, and loan exposures to differ across subsets of loans in the portfolio. The results turned out to be very similar to the homogeneous case. The tail thickness of credit losses is determined by that part of the portfolio that has the heaviest idiosyncratic tail. In particular, the credit loss tail shape of a heterogeneous portfolio is the same as that of a homogeneous portfolio consisting solely of the bonds with the heaviest idiosyncratic tail.

We also investigated the effect of changes in credit quality as measures by the magnitude of the default threshold. We found that portfolio quality affects credit loss tail behaviour rather differently in the CreditMetrics compared to the CreditRisk$^+$ framework. Whereas the portfolio quality directly enters the tail index in a CreditRisk$^+$ setting, it only influences CreditMetrics portfolio losses indirectly via the so-called slowly varying function.

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Appendix: Proofs

Proof of Theorem 1: Consider the filtration defined by the σ-fields $\mathcal{F}_n = \sigma(f, \varepsilon_1, \ldots, \varepsilon_n)$ and the process M given by $M_n = \sum_{j=1}^n \xi_j$, with $\xi_j = 1_{\{S_j < s^*\}} - P(S_j < s^* | f)$. We show that M is a zero mean martingale with respect to this filtration. To that end we have to show that $E[\xi_n | \mathcal{F}_{n-1}] = 0$ for all $n \geq 1$. We compute, using that $P(S_n < s^* | f)$ is $\mathcal{F}_{n-1}$-measurable

$$
\begin{aligned}
E[\xi_n | \mathcal{F}_{n-1}] &= E\left[1_{\{g(f, \varepsilon_n) < s^*\}} | \sigma(f, \varepsilon_1, \ldots, \varepsilon_{n-1})\right] - P(S_n < s^* | f) \\
&= P(g(f, \varepsilon_n) < s^*) | \sigma(f, \varepsilon_1, \ldots, \varepsilon_{n-1}) - P(g(f, \varepsilon_n) < s^* | f)
\end{aligned}
$$

Due to the assumed independence of f, ε_1, ε_2, ..., the last two conditional probabilities are the same and the martingale property follows.

Moreover, $P(S_n < s^* | f) = P(g(f, \varepsilon_n) < s^* | f) = P(\varepsilon_n < \tilde{\varepsilon}(f, s^*) | f)$. Invoking the assumed independence once more we can rewrite this as $F(\tilde{\varepsilon}(f, s^*))$. Notice that this is independent of n and we therefore define $C := P(S_n < s^* | f) = F(\tilde{\varepsilon}(f, s^*))$. In the same vein one can compute var $(\xi_n | \mathcal{F}_{n-1}) = C(1-C) < 1$ a.s. Hence, it follows from the strong law for martingales (Williams, 1991, Theorem 12.14) that $\frac{M_n}{n} \to 0$ a.s. Finally, since it holds that

$$
C_n = \frac{M_n}{n} + C
$$

the a.s. convergence of C_n to C follows.

Proof of Theorem 2: If H is the distribution functions of C, then we have for all $c \in (0, 1)$ that $\bar{H}(c) = 1 - H(c) = G(\bar{f}(s^*, F^{-1}(c)))$, since $C = F(\tilde{\varepsilon}(f, s^*))$ and

$$
\begin{aligned}
\bar{H}(c) &= P[C > c] = P[F(\tilde{\varepsilon}(f, s^*)) > c] = P\left[\tilde{\varepsilon}(f, s^*) > F^{-1}(c)\right] \\
&= P\left[f < \bar{f}(s^*, F^{-1}(c))\right] = G(\bar{f}(s^*, F^{-1}(c)))
\end{aligned}
$$

The result follows from the composition rule for regular varying functions, see Bingham *et al.* (1987), Proposition 1.5.7. It states that the composition $R_1 \circ R_2$ of two regularly varying functions (at infinity) R_1 and R_2 with indices θ_1 and θ_2 is regularly varying with index $\theta_1\theta_2$ if $R_2(x) \to \infty$ as $x \to \infty$. To apply this proposition (two times) to the function $\bar{H}$ one switches to the auxiliary function h defined by $h(x) = G(\bar{f}(s^*, F^{-1}(1 - 1/x)))$. First we verify that $x \to F^{-1}(1 - 1/x)))$ is regularly varying at infinity. It follows from Assumption 1 that F^{-1} is well defined and that $\lim_{x\to\infty} F^{-1}(1 - 1/x))) = \infty$. From Theorem 1.5.12 of Bingham *et al.* (1987) we obtain that this function is regularly varying with exponent $-1/v_1$. A twofold application of the composition rule is now justified under Assumptions 1 and 2A.

Proof of Theorem 3: In the course of the proof we need certain properties of functions belonging to the classes $\mathcal{M}_a(\theta)$. We give these properties first. The following statements parallel Seneta (1976), pp. 18, 19, and Bingham *et al.* (1987), Proposition 1.5.7. If χ is increasing (with $\theta > 0$) and $\chi \in \mathcal{M}_\infty(\theta)$, then $\chi^{-1} \in \mathcal{M}_\infty(1/\theta)$. Similarly, if χ is decreasing (with $\theta < 0$), then $\chi^{-1} \in \mathcal{M}_0(1/\theta)$. We also have that $\chi \in \mathcal{M}_a(\theta_1)$ and $\psi \in \mathcal{M}_b(\theta_2)$ with $\lim_{x\to b} \psi(x) = a$ implies that $\chi \circ \psi \in \mathcal{M}_b(\theta_1\theta_2)$. Furthermore, if $\chi \in \mathcal{M}_a(\theta)$ then $\chi^\alpha \in \mathcal{M}_a(\alpha\theta)$ and if $\chi \in \mathcal{M}_a(\theta_1)$ and $\psi \in \mathcal{M}_a(\theta_2)$, then $\chi\psi \in \mathcal{M}_a(\theta_1 + \theta_2)$.

We now start proving the theorem. The function ϕ defined by $\phi(x) = \log \bar{F}(x)$ (here $\bar{F} = 1 - F$) belongs to $\mathcal{M}_\infty(v_2)$. It then follows that ϕ^{-1} belongs to $\mathcal{M}_{-\infty}(1/v_2)$. For $y \to -\infty$ we have

$$\phi^{-1}(y) = \left(\frac{y}{v_1}\right)^{1/v_2} L_{\phi^{-1}}(y)$$

As a consequence, we can write

$$\bar{F}^{-1}(t) = F^{-1}(1-t) = \left(\frac{\log t}{v_1}\right)^{1/v_2} L_{F^{-1}}(t)$$

where $L_{F^{-1}}(t) = L_{\phi^{-1}}(\log t)$ is so that $\lim_{t\to 0} L_{F^{-1}}(t) = 1$, since $\lim_{x\to\infty} \varepsilon(x) = 0$. Using the assumptions on $\tilde{f}$, we can write

$$\tilde{f}\left(s^*, F^{-1}(1-t)\right) = -\zeta_2^{1/\mu_2}\left(\frac{\log t}{v_1}\right)^{1/\mu_2} L_{F^{-1}}^{v_2/\mu_2}(t)(1 + \tilde{\eta}(t))$$

with $\tilde{\eta}(t) = \eta\left(F^{-1}(1-t)\right) = \eta\left(\left(\frac{\log t}{v_1}\right)^{1/v_2} L_{F^{-1}}(t)\right)$. Notice that $\tilde{\eta}(t) \to 0$ for $t \to 0$. As a result we get

$$\bar{H}(1-t) = \exp(\alpha k(t) \log t)$$

with

$$\alpha = \frac{\mu_1\zeta_2}{v_1}$$

$$k(t) = L_{F^{-1}}(t)^{v_2}(1 + \tilde{\eta}(t))^{\mu_2}\left(1 + \tilde{\xi}(t)\right)$$

and

$$\tilde{\xi}(t) = \xi\left(\tilde{f}\left(s^*, F^{-1}(1-t)\right)\right)$$

Notice that also $\tilde{\xi}(t) \to 0$ for $t \to 0$.

In order to have that $t \mapsto \bar{H}(1-t)$ is regularly varying at zero with coefficient α_1 we have to prove that for $t \to 0$

$$\log \bar{H}(1-xt) - \log \bar{H}(1-t) \to \alpha \log x$$

which amounts to

$$(k(tx) - k(t)) \log t + k(tx) \log x \to \log x$$

Hence, we verify that

$$\lim_{t\to 0} k(tx) = 1$$

and

$$\lim_{t\to 0}(k(tx) - k(t)) \log t = 0 \tag{A1}$$

The first limit is obvious from the definition of k and the convergence of η and ξ. The second limit will be treated by using properties of functions belonging to the relevant classes $\mathcal{M}(\theta)$. We make the substitutions $y = \log t$ and $z = \log x$. Putting $\ell(y) = k(e^y)$ we rewrite equation (A1) as

$$\lim_{y\to -\infty} y(\ell(y+z) - \ell(y)) = 0 \tag{A2}$$

Observe that

$$\ell(y) = L_{\phi^{-1}}(y)^{\nu_2}\left(1 + \eta\left(\phi^{-1}(y)\right)\right)^{\mu_2}\left(1 + \xi\left(\tilde{f}\left(s^*, \phi^{-1}(y)\right)\right)\right)$$

Since $\phi^{-1}(y) \to \infty$ when $y \to -\infty$ we can draw the following conclusions, using the properties listed at the beginning of the proof. The functions $L_{\phi^{-1}}$, $1 + \eta \circ \phi^{-1}$ and $1 + \xi\left(\tilde{f}\left(s^*, \phi^{-1}(\cdot)\right)\right)$ belong to $\mathcal{M}_{-\infty}(0)$. Hence we find that also $\ell \in \mathcal{M}_{-\infty}(0)$. Next we invoke the mean value theorem to write $\ell(y+z) - \ell(y) = z\ell'(y+z^*)$ for some z^* between zero and z. But then $y(\ell(y+z) - \ell(y)) = z\frac{y}{y+z^*}\frac{(y+z^*)\ell'(y+z^*)}{\ell(y+z^*)}\ell(y+z^*) \to 0$. This shows that (A2) is valid and concludes the proof of the theorem.

Proof of Theorem 4: Note that

$$P[C > c] = P\left[\sum_{i=1}^{m} \lambda_i\left[1 - F_i\left(\tilde{\varepsilon}_i\left(f, s_i^*\right)\right)\right] < 1 - c\right] \tag{A3}$$

Notice that in (A3) we are interested in extreme events that are determined in terms of f. If c tends to one, then the event $\{C > c\}$ can alternatively be expressed as $\{f < l\}$, where l tends to $-\infty$. What we want to show is that the tail behaviour of C, which is determined by the tail behaviour of $\sum_{i=1}^{m} \lambda_i\left[1 - F_i\left(\tilde{\varepsilon}_i\left(f, s_i^*\right)\right)\right]$, is essentially determined by that of $1 - F_\iota\left(\tilde{\varepsilon}_\iota\left(f, s_\iota^*\right)\right)$. More precisely, we show that for $c \uparrow 1$, we have $P(C > c) \sim P\left(1 - F_\iota\left(\tilde{\varepsilon}_\iota\left(f, s_\iota^*\right)\right) < \frac{1-c}{K}\right)$.

We use the following auxiliary result. Let h_1 and h_2 be two (measurable) increasing functions, such that $\lim_{x\to\infty} h_1(x)/h_2(x) = 1$. Let X be a random variable with distribution function F^X that is such that for all x one has $F^X(x) < 1$. Consider the random variables $h_1(X)$ and $h_2(X)$. If $h_2(X)$ has a regularly varying tail at infinity, then also $h_1(X)$ has a regularly varying tail, with the same index as $h_2(X)$. A similar statement holds for left tails.

Proof of the auxiliary result: Fix $\delta > 0$ and choose x_0 such that $x > x_0$ implies $\left|\frac{h_1(x)}{h_2(x)} - 1\right| < \delta$. Consider then $P(h_1(X) > u)$, where u is sufficiently big, such that we must have $X > x_0$. Then we have $P(h_1(X) > u) \le P\left(h_2(X) > \frac{u}{1+\delta}\right)$ and $P(h_1(X) > u) \ge P\left(h_2(X) > \frac{u}{1-\delta}\right)$. Hence we have the double inequality

$$\frac{P\left(h_2(X) > \frac{u}{1-\delta}\right)}{P(h_2(X) > u)} \le \frac{P(h_1(X) > u)}{P(h_2(X) > u)} \le \frac{P\left(h_2(X) > \frac{u}{1+\delta}\right)}{P(h_2(X) > u)}$$

Let $\theta < 0$ be the index of regular variation of the right hand tail of $h_2(X)$. Using that $h_2(X)$ has a regularly varying right tail, we then obtain that

$$\limsup_{u\to\infty} \frac{P(h_1(X) > u)}{P(h_2(X) > u)} \leq (1+\delta)^{\theta}$$

and

$$\liminf_{u\to\infty} \frac{P(h_1(X) > u)}{P(h_2(X) > u)} \geq (1-\delta)^{\theta}$$

Since this is true for all $\sigma > 0$, we conclude that

$$\lim_{u\to\infty} \frac{P(h_1(X) > u)}{P(h_2(X) > u)} = 1$$

But then

$$\frac{P(h_1(X) > ut)}{P(h_1(X) > u)} = \frac{P(h_1(X) > ut)}{P(h_2(X) > ut)} \cdot \frac{P(h_2(X) > u)}{P(h_1(X) > u)} \cdot \frac{P(h_2(X) > ut)}{P(h_2(X) > u)}$$

converges to t^{θ} as $u \to \infty$.

The auxiliary result can now be used with f instead of X and letting $\sum_{i=1}^{m} \lambda_i \left[1 - F_i\left(\tilde{\varepsilon}_i\left(f, s_i^*\right)\right)\right]$ take the role of $h_1(X)$ and $K\left(1 - F_\iota\left(\tilde{\varepsilon}_\iota\left(f, s_\iota^*\right)\right)\right)$ the role of $h_2(X)$.

Proof of Theorem 5: Similar to the proof of Theorem 4.

Proof of Theorem 6: Using the fact that for $x \downarrow -\infty$ we have $\Phi(x) = |x|^{-1}\phi(x)\left(1 + O\left(|x|^{-2}\right)\right)$, see (26.2.13) in Abramowitz and Stegun (1970), we obtain for $\xi \downarrow 0$ that

$$\begin{aligned} P[C > 1-\xi] &= \Phi\left(\frac{s + \Phi^{-1}(\xi)\sqrt{1-\rho^2}}{\rho}\right) \\ &\sim \frac{\phi\left(\frac{s+\Phi^{-1}(\xi)\sqrt{1-\rho^2}}{\rho}\right)}{\frac{\left|s+\Phi^{-1}(\xi)\sqrt{1-\rho^2}\right|}{\rho}} \\ &= \exp\left(-\frac{s^2}{2\rho^2} - \frac{s\Phi^{-1}(\xi)\sqrt{1-\rho^2}}{2\rho^2}\right)\left[\frac{\phi\left(\Phi^{-1}(\xi)\right)}{\left|\Phi^{-1}(\xi)\right|}\right]^{\frac{1-\rho^2}{\rho^2}} \frac{\left|\Phi^{-1}(\xi)\right|^{\left(1-\rho^2\right)/\rho^2}}{\left|\frac{s+\Phi^{-1}(\xi)\sqrt{1-\rho^2}}{\rho}\right|} \\ &\sim \exp\left(-\frac{s^2}{2\rho^2} - \frac{s\Phi^{-1}(\xi)\sqrt{1-\rho^2}}{2\rho^2}\right)[\xi]^{\frac{1-\rho^2}{\rho^2}} \frac{\left|\Phi^{-1}(\xi)\right|^{\left(1-\rho^2\right)/\rho^2}}{\frac{\left|s+\Phi^{-1}(\xi)\sqrt{1-\rho^2}\right|}{\rho}} \end{aligned} \tag{A4}$$

Let $\hat{\Phi}(x) = \phi(x)/|x|$ for $x < 0$. Then $\hat{\Phi}$ is strictly increasing on $(-\infty, 0)$ and

$$\hat{\Phi}(x) = \frac{\phi(x)}{|x|} = \frac{1}{\sqrt{2\pi x^2 \exp(x^2)}} \Leftrightarrow \left(\sqrt{2\pi}\,\hat{\Phi}(x)\right)^{-2} = x^2 \exp\left(x^2\right)$$

Replacing $\hat{\Phi}(x)$ by ξ, x by $\hat{\Phi}^{-1}(\xi)$, and noting that the Lambert-W function $W(\cdot)$ (Corless *et al.*, 1996), is defined as

$$W(x)\cdot \exp[W(x)] = x$$

we obtain directly that

$$\left(\sqrt{2\pi}\,\hat{\Phi}(x)\right)^{-2} = x^2 \exp\left(x^2\right) \;\Leftrightarrow\; x^2 = W\left(\left(\sqrt{2\pi}\,\hat{\Phi}(x)\right)^{-2}\right). \;\Leftrightarrow$$

$$\hat{\Phi}^{-1}(\xi) = -\sqrt{W\left(\left(\sqrt{2\pi}\,\xi\right)^{-2}\right)}$$

The Lambert W function satisfies the asymptotic approximation (for $x \to \infty$)

$$W(x) = \ln(x) - \ln(\ln(x)) + o(\ln(\ln(x)))$$

(Corless *et al.*, 1996), such that

$$\hat{\Phi}^{-1}(\xi) \overset{\xi\downarrow 0}{=} -\sqrt{-\ln\left(2\pi\xi^2\right)} \tag{A5}$$

Substituting $\Phi^{-1}(\xi)$ in (A4) by (A5), we obtain the desired result.